Homework #1

Natalie Pueyo Svoboda, 997466498

January 13, 2018

1 First Problem

Solution. To prove that $f(n) = \mathcal{O}(n^2)$, if $f(n) = 3n^2 + 2n + 4$, I have to prove that there are constants c and M such that $\forall n \geq M$, $f(n) \leq c \cdot g(n)$, where $g(n) = n^2$. In other words, there is a constant, c that when multiplied $g(n)$, will allow $g(n)$ to always overtake $f(n)$ $\forall n \geq M$. In this case, $f(n) = 3n^2 + 2n + 4 \leq c \cdot n^2$.
To make the proof easier, I chose $M = 1$ such that $n \geq 1$. I can then use this definition to find a term greater than each term in $f(n)$ so that $f(n) = 3n^2 + 2n + 4 \leq c \cdot n^2$. The resulting terms have to deal with the constants in $f(n)$.
Because n was defined such that $n \geq 1$, we can then multiply both sides by n to show that $n^2 \geq n$ resulting in

$$n^2 \geq n \geq 1$$

by multiplying each term in the inequality by the constant we want to remove, we can show that:

$$2n^2 \geq 2n \geq 2$$

and

$$4n^2 \geq 4n \geq 4$$

which takes care of the $2n$ and 4 constants respectivelly. To find a value greater than $f(n)$'s first term, we use the definition of the $\geq$ operator: $3n^2 \geq 3n^2$ by definition.
This leads to the following inequality:

$$3n^2 + 2n + 4 \leq 3n^2 + 2n^2 + 4n^2$$

Each term on the right is greater than or equal to its corresponding term on the left which makes the sum of all the right terms greater than $f(n)$.

$$3n^2 + 2n + 4 \leq 9n^2$$

This means that if $g(n)$ is multiplied by the constant $c = 9$, after $n \leq M$, where $M = 1$, $f(n) \leq c \cdot g(n)$ proving that $f(n) = \mathcal{O}(n^2)$.

2 Second Problem

Solution. If $f(n) = log(n)$ and $f(n) = \mathcal{O}(g(n))$ it can be proven that $g(n) = n$ if $lim_{n\to\infty}\frac{log(n)}{n} = c$ for some constant $c \geq 0$. Using L'Hopital's rule

$$lim_{n\to\infty}\frac{f(n)}{g(n)} = \frac{f'(n)}{g'(n)}$$

such that

$$\begin{aligned} lim_{n\to\infty}\frac{log(n)}{n} &= lim_{n\to\infty}\frac{f'(n)}{g'(n)} = c \\ &= lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{1} = c \\ &= lim_{n\to\infty}\frac{1}{n} = c \\ &= 0 = c \end{aligned}$$

Which satisfies the condition that $c \geq 0$ proving that $log(n) = \mathcal{O}(n)$. This proof was done with the assumption that $log(n)$ was $ln(n)$. However, it is also true for other bases because the addition of a base just adds a constant which is ignored in Big-O notation. This can also be used to prove that $n^k log(n) = \mathcal{O}(n^{k+1})$ for any constant k.

$$\begin{aligned} lim_{n\to\infty}\frac{n^k log(n)}{n^{k+1}} &= lim_{n\to\infty}\frac{f'(n)}{g'(n)} = c \\ &= lim_{n\to\infty}\frac{n^{k-1}(k\cdot log(n)+1}{(k+1)n^k} = c \\ &= lim_{n\to\infty}\frac{k\cdot log(n)+1}{n(k+1)} = c \\ &= \frac{1}{k+1}lim_{n\to\infty}\frac{k\cdot log(n)+1}{n} = c \\ &= lim_{n\to\infty}\frac{f'(n)}{g'(n)} \\ &= \frac{1}{k+1}lim_{n\to\infty}\frac{k}{n}\cdot\frac{1}{1} = c \\ &= \frac{k}{k+1}lim_{n\to\infty}\frac{1}{n} = c \\ &= \frac{k}{k+1}\cdot 0 = 0 = c \end{aligned}$$

This also satisfies the condition that $c \geq 0$ proving that $n^k log(n) = \mathcal{O}(n^{k+1})$. For this second proof it was necessary to use L'Hopital's Rule a second time to make the proof possible.

3 Third Problem

Solution. The logarithmic identity to change the base of a logarithm is

$$log_a x = \frac{log_b x}{log_b a}$$

Using this identity we can prove that if we have two constants a and b, and $f(n) = log_a n$, then $log_a n = \mathcal{O}(log_b n)$ and that $log_b n = \mathcal{O}(log_a n)$. By using the identity we can show that

$$log_a n = \mathcal{O}\left(\frac{log_b n}{log_b a}\right)$$

and since $log_b a$ is a constant it gets discarded from the Big-O notation. This results in

$$log_a n = \mathcal{O}(log_b n)$$

The same excercise can be done for $f(n) = log_b n$ where

$$log_b n = \mathcal{O}\left(\frac{log_a n}{log_a b}\right)$$

where again we discard the constant $log_a b$ resulting in

$$log_b n = \mathcal{O}(log_a n)$$

4 Fourth Problem

Solution. If we imply that $f_i = \mathcal{O}(f_{i+1})$ for all $1 \leq i \leq 5$

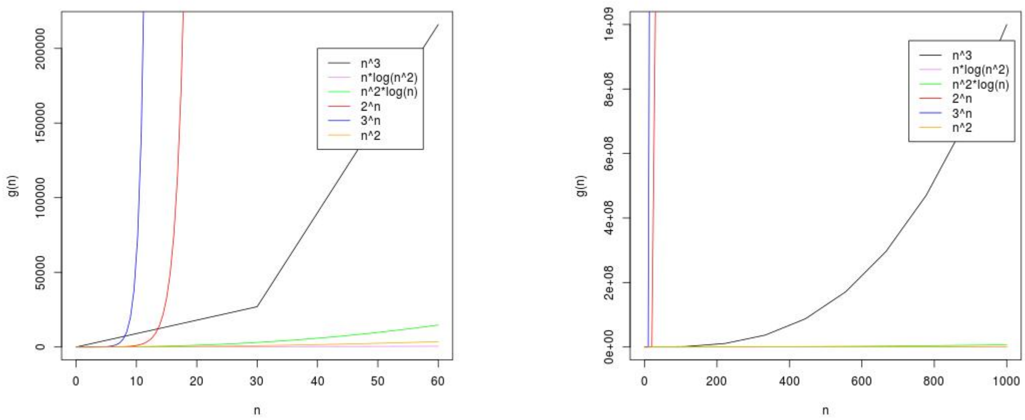
$f(n)$	$\mathcal{O}(g(n))$
$n^3 + 2n + 1$	$\mathcal{O}(n^3)$
$nlog(n^2)$	$\mathcal{O}(n \cdot log(n^2))$
$n^2 log(n)$	$\mathcal{O}(n^2 log(n))$
2^n	$\mathcal{O}(2^n)$
3^n	$\mathcal{O}(3^n)$
$1023n^2 + 2n + 45$	$\mathcal{O}(n^2)$

The $\mathcal{O}(g(n))$ of each given formula was plotted in R to determine which ones grew faster asymptotically. This would make it easy to rank them such that $f_i = \mathcal{O}(f_{i+1})$ for all $1 \leq i \leq 6$. Figure 4.1 shows how all 6 equations act as n grows larger.

With the plots the equations can be ranked according to the speed of their growth as:

$$n \cdot log(n^2), n^2, n^2 \cdot log(n), n^3, 2^n, 3^n$$

Making $f_1 = n \cdot log(n^2)$, $f_2 = 1023n^2 + 2n + 45$, $f_3 = n^2 \cdot log(n)$, $f_4 = n^3 + 2n + 1$, $f_5 = 2^n$, and $f_6 = 3^n$.



(a) $g(n)$ plots when $n = 80$ (b) $g(n)$ plots when $n = 1000$

Figure 4.1: Plots of $\mathcal{O}(g(n))$ for the various $f(n)$ equations as n grows larger

5 Fifth Problem

Solution. To get the Big-O of a sum, one has to first find the resulting formula of the sum. The sum's for this problem is: $\sum_{i=0}^{n} i$ which expanded gives:

$$S = \sum_{i=0}^{n} i = 1 + 2 + \ldots + (n-1) + n$$

To solve for the formula,

$$\begin{aligned} 2 \cdot S = 2 \cdot \sum_{i=0}^{n} i &= (1 + 2 + \ldots + (n-1) + n) + (1 + 2 + \ldots + (n-1) + n) \\ &= (1 + 2 + \ldots + (n-1) + n) + (n + (n-1) + \ldots + 2 + 1) \\ &= [1 + n] + [2 + (n-1)] + ... + [(n-1) + 2] + [n + 1] \\ &= n \cdot (n+1) \end{aligned}$$

When solving for S, the result is $S = \sum_{i=0}^{n} i = \frac{n \cdot (n+1)}{2}$. This can then be used to prove that $\sum_{i=0}^{n} i = \mathcal{O}(n^2)$ because the dominant term of $\frac{n \cdot (n+1)}{2}$ is n^2 from simplifying S to $\frac{1}{2}(n^2 + n)$.

6 Sixth Problem

Solution. For any positive integer n and assuming that $\sum_{i=0}^{n} 3^i \leq 3^{n+1}$,

1. Base step: $\sum_{i=0}^{0} 3^i = 3^0 \leq 3^{0+1} = 3^1$ is true.

2. Inductive step: assuming $\sum_{i=0}^{n+1} 3^i = \sum_{i=0}^{n} 3^i + 3^{n+1}$

$$\begin{aligned}\sum_{i=0}^{n} 3^i + 3^{n+1} &\le 3^{n+1} + 3^{n+1} \\ &\le 2 \cdot 3^{n+1} \\ &\le 3^{(n+1)+1}\end{aligned}$$

so the inductive step holds true as well, so by induction $\sum_{i=0}^{n} 3^i \le 3^{n+1}$ for any positive integer n.